Attractor switching in neuron networks and spatiotemporal filters for motion processing

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Attractor switching in neuron networks

Chapter 2

Pulse coupled oscillator network with delay

Abstract

This chapter formulates a mathematical description for a system of pulse coupled oscillators with delay. The purpose is to introduce notations and to discuss some general results concerning a pulse-coupled system. In particular, a symbolic representation of the dynamics, called event representation, is described which is used in later chapters to prove several results.

In a system of pulse coupled oscillators, an individual oscillator is described by a phase variable that is a function of time. The phase variable varies between 0 and 1 and lives in $\mathbb{T}^1$ (or $\mathbb{S}^1$). When the phase of an oscillator reaches the threshold 1, the oscillator fires and its phase is reset to zero. Consequently, a pulse is sent to all the other oscillators which is received after a time delay τ. The interactions between the oscillators occur through a linear monotonic function called *pulse response function*.

The goal of this chapter is to present a mathematical setting of the system and is organized as follows. We begin with the definition of pulse response function, the dynamics of the system, its state space and evolution operator in section 2.1.1. In section 2.1.2 a specific pulse response function called Mirollo Strogatz function is discussed. Then in section 2.1.3 we introduce a metric on the state space and study the continuity of the evolution operator with respect to the defined metric. Finally, in section 2.2, two other representations of the dynamics called *past-firings representation* and *event representation* are described. Certain general properties of the metric are discussed in the appendix of this chapter. The representations as well as the properties of the metric are used in later chapters to prove several results.

2.1 Setting of the problem

The system studied in this chapter is a delay system (Diekmann et al. 1995). The state space of such systems is an appropriate space $\mathcal{P}_\tau^n$ of functions (see definition 1) defined on the interval $(-\tau, 0]$, where $\tau > 0$ is the delay of the system, and taking values in an n-dimensional manifold N. The state space thus is infinite dimensional. In our case, points in N represent the phases of the n coupled oscillators, which implies that $N = \mathbb{T}^n$, the n-dimensional torus.

For a given $\phi \in \mathcal{P}_\tau^n$ and for each $t \in (-\tau, 0]$, $\phi(t) \in N$ represents the phases of the oscillators at time t. Using the dynamics of the system, ϕ can be *extended* to a unique function $\phi^+ : (-\tau, +\infty) \to N$, such that $\phi^+(t) = \phi(t)$ for $t \in (-\tau, 0]$ and $\phi^+(t) \in N$ represents the phases of the oscillators at any time $t \geq -\tau$. Then the *evolution operator* $\Phi^t : \mathcal{P}_\tau^n \to \mathcal{P}_\tau^n$ is defined by $\Phi^t(\phi)(s) = \phi^t(s) = \phi^+(t+s)$ for any $t \geq 0$ and $s \in (-\tau, 0]$. In other words, the evolution operator maps the initial state $\phi = \phi^0$ to the state ϕ^t of the system at time t. The latter is the restriction of ϕ^+ in $(t-\tau, t]$ shifted back to the interval $(-\tau, 0]$.

2.1.1 Definition of the dynamics

We now specialize the above notions of the theory of delay equations to the current setting. In this section we follow closely (Ashwin and Timme 2005a).

Definition 1 (State space, cf. (Ashwin and Timme 2005a)). *The state space $\mathcal{P}_\tau^n$ of the system of n pulse coupled oscillators with delay $\tau > 0$ is the space of* phase history functions

$$\phi : (-\tau, 0] \to \mathbb{T}^n : s \mapsto \phi(s) = (\phi_1(s), \dots, \phi_n(s)),$$

that satisfy the following conditions:

1. *Each ϕ_i is upper-semicontinuous, i.e., $\phi_i(s^+) := \lim_{t \to s^+} \phi_i(t) = \phi_i(s)$ and $\phi_i(s^-) := \lim_{t \to s^-} \phi_i(t) \leq \phi_i(s)$ for all $s \in (-\tau, 0]$.*
2. *Each ϕ_i is only discontinuous at a finite (or empty) set $S_i = \{s_{i,1}, \dots, s_{i,k_i}\} \subset (-\tau, 0]$ with $k_i \in \mathbb{N}$ and $s_{i,1} > s_{i,2} > \dots > s_{i,k_i}$.*
3. *$d\phi_i(s)/ds = 1$ for $s \notin S_i$.*

The coupling between the n oscillators is defined using the *pulse response function.*

Definition 2 (Pulse response function, cf. (Ashwin and Timme 2005a)). *A* pulse response function *is a map*

$$V : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto V(\theta, \varepsilon), \tag{2.1}$$

that satisfies the following conditions:

1. *V is smooth on $(\mathbb{T} \setminus \{0\}) \times \mathbb{R}_+$.*
2. *$\partial V(\theta, \varepsilon)/\partial\theta > 0$ on $(\mathbb{T} \setminus \{0\}) \times (\mathbb{R}_+ \setminus \{0\})$.*

3. *$\partial V(\theta, \varepsilon)/\partial\varepsilon > 0$ on $\mathbb{T} \times \mathbb{R}_+$.*

4. *$V(\theta, 0) = 0$ for all $\theta \in \mathbb{T}$.*

5. *$0 < V(0, \varepsilon) < 1$ for all $\varepsilon \in (0, 1)$.*

6. *H, given by (2.4), satisfies*

$$H_m(\theta) = H_1 \circ H_{m-1}(\theta) = \overbrace{H_1 \circ \cdots \circ H_1}^{m\text{-}times}(\theta). \tag{2.2}$$

Notice that in the above definition $\partial V/\partial\theta > 0$, therefore V cannot be smooth everywhere on $\mathbb{T}$. This is reflected in condition (i) of the definition. The pulse response function depends on the parameter $\varepsilon \geq 0$, called *coupling strength.* As a shorthand notation we introduce

$$V_m(\theta) = V(\theta, m\hat{\varepsilon}), \text{ for } m = 1, 2, 3, \ldots, \tag{2.3}$$

where $\hat{\varepsilon} = \varepsilon/(n-1)$. Given a pulse response function V we also define

$$H : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto H(\theta, \varepsilon) = \theta + V(\theta, \varepsilon), \tag{2.4}$$

and

$$H_m(\theta) = H(\theta, m\hat{\varepsilon}), \text{ for } m = 1, 2, 3, \ldots. \tag{2.5}$$

Definition 3 (Dynamics, cf. (Ashwin and Timme 2005a)). *A system of n pulse coupled oscillators with delay is a quadruple $\mathcal{D} = (n, V, \varepsilon, \tau)$, where V is as in definition 2, $\varepsilon \geq 0$ and $\tau \geq 0$. Given a system $\mathcal{D}$ and an initial state $\phi \in \mathcal{P}^n_\tau$, we extend ϕ to a function $\phi^+ : (-\tau, +\infty) \to \mathbb{T}^n$ using the following rules:*

1. *$\phi^+(t) = \phi(t)$ for $t \in (-\tau, 0]$.*

2. *$d\phi_i^+(t)/dt = 1$ for $t \geq 0$, if $\phi_j^+(t-\tau) \neq 0 \pmod{\mathbb{Z}}$ for all $j \neq i$.*

3. *$\phi_i^+(t) = \min\{1, H_m(\phi_i^+(t^-))\} \pmod{\mathbb{Z}}$, if there are $j_1, \ldots, j_m \neq i$ such that $\phi_{j_k}^+(t - \tau) = 0 \pmod{\mathbb{Z}}$ for all $k = 1, \ldots, m$.*

The dynamics described in definition 3 can be interpreted in the following way. The phase ϕ_i of each oscillator O_i, $i = 1, \ldots, n$, increases linearly. When the phase reaches the value $1 = 0 \pmod{\mathbb{Z}}$, then the oscillator O_i *fires* and all the other oscillators O_j, $j \neq i$ receive a *pulse* after a time *delay* τ. In general, an oscillator O_j may receive m simultaneous pulses at time t if m oscillators $O_{i_1}, \ldots, O_{i_m}$ have fired simultaneously at time $t - \tau$. Then the phase of O_j is increased to $H(u_j, m\hat{\varepsilon}) = H_m(u_j)$ where $u_j = \phi_j^+(t^-)$, unless the pulse causes the oscillator to fire and then the phase becomes exactly 1.

The *evolution operator* Φ^t for $t \geq 0$ is then defined by

$$\Phi^t : \mathcal{P}_\tau^n \to \mathcal{P}_\tau^n : \phi \mapsto \Phi^t(\phi) = \phi^t = \phi^+|_{(t-\tau,t]} \circ T_t, \tag{2.6}$$

where T_t is the shift $s \mapsto s + t$ and the *positive semiorbit* of $\phi \in \mathcal{P}_\tau^n$ is given by

$$\mathcal{O}_+(\phi) = \{\Phi^t(\phi) : t \geq 0\}. \tag{2.7}$$

Proposition 1. *The evolution operator Φ^t is well defined.*

Proof. From definition 3 it follows that the extended function ϕ^+ can be determined for all $t \geq 0$ and all $\phi \in \mathcal{P}_\tau^n$, given $\mathcal{D}$ and $\phi \in \mathcal{P}_\tau^n$. The only question is whether $\phi^t \in \mathcal{P}_\tau^n$ for all $t \geq 0$. First we show that ϕ^t, $t \geq 0$ is discontinuous at a finite set. Note that, by definition 1, each component ϕ_i of ϕ has only a finite number k_i of discontinuities in $(-\tau, 0]$. Therefore, $\phi_i(0) < \phi_i(-\tau) + k_i + \tau$, since the phase ϕ_i increases linearly (outside discontinuities) and each discontinuity induces an increase of ϕ_i that is less than 1. This implies that $\phi_i(s) = 0 \pmod{\mathbb{Z}}$ in a finite set $\{s_{i,1}, \ldots, s_{i,\ell_i}\} \subset (-\tau, 0]$ with ℓ_i elements. Then, the number of discontinuities of ϕ_j^+, in $(0, \tau]$ (and hence of ϕ_j^τ) also is finite for all $j = 1, \ldots, n$. This follows from the fact that the number of discontinuities of ϕ_j^+ in $(0, \tau]$ is less than or equal to the number of firings of all the ϕ_i in $(-\tau, 0]$ for $i = 1, \ldots, n$ and $i \neq j$, therefore it is less than or equal to $\sum_{i=1}^n \ell_i$. This shows that advancing time by τ the number of discontinuities remains finite. It follows by induction, that the number of discontinuities of ϕ_i^+ in any interval $((m-1)\tau, m\tau]$, $m \in \mathbb{N}$ is finite for all $i = 1, \ldots, n$. Thus, the number of discontinuities of ϕ_i^+ in any interval $(t - \tau, t]$, $t \geq 0$ (and hence of ϕ_i^t) is finite for all $i = 1, \ldots, n$. The facts that ϕ^t is upper-semicontinuous and $d\phi_i^t/ds = 1$ (outside discontinuities) are a direct consequence of properties (iii) and (ii) respectively of definition 3. □

For a given system $\mathcal{D} = (n, V, \varepsilon, \tau)$, the *accessible state space* is $\mathcal{P}_\mathcal{D} = \Phi^\tau(\mathcal{P}_\tau^n)$. In other words, $\phi \in \mathcal{P}_\mathcal{D}$ if there is a state $\psi \in \mathcal{P}_\tau^n$ such that $\Phi^\tau(\psi) = \phi$, i.e., $\mathcal{P}_\mathcal{D}$ includes only those states that are dynamically accessible. From now on, we restrict our attention to $\mathcal{P}_\mathcal{D}$.

2.1.2 The Mirollo-Strogatz model

A pulse response function V that satisfies all the requirements of definition 2 is provided by the Mirollo-Strogatz model (Mirollo and Strogatz 1990) where the pulse response function is

$$V_{\mathrm{MS}}(\theta, \varepsilon) = f^{-1}(f(\theta) + \varepsilon) - \theta, \tag{2.8}$$

and f is a function which is concave down ($f'' < 0$) and monotonically increasing ($f' > 0$). Moreover, $f(0) = 0$ and $f(1) = 1$. A concrete example is given by

$$f_b(\theta) = \frac{1}{b} \ln(1 + (e^b - 1)\theta). \tag{2.9}$$

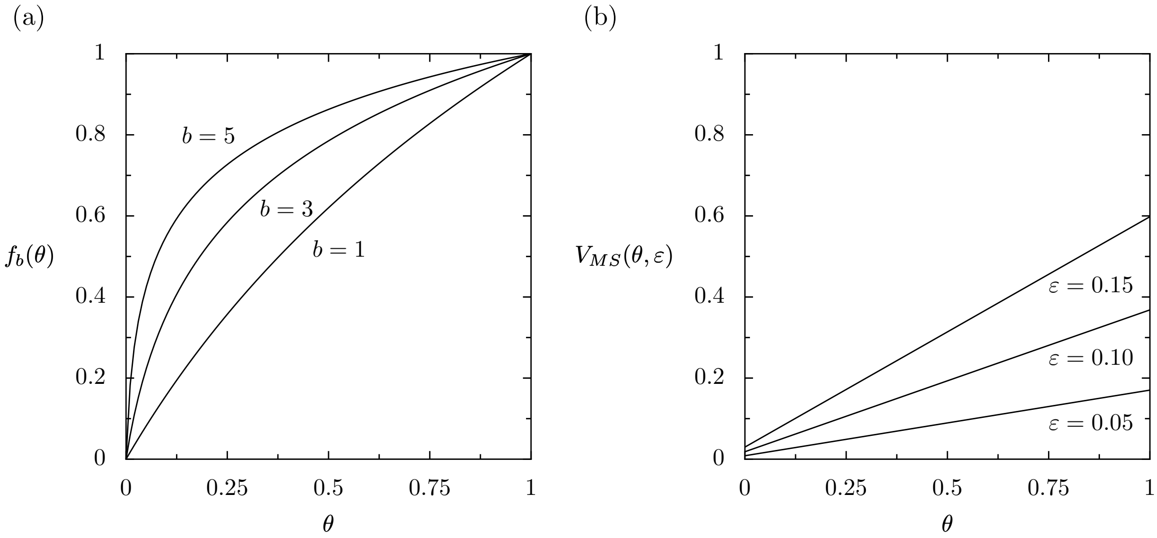

Figure 2.1: *(a) Graph of f_b (2.9) as a function of θ for different values of b. (b) Graph of V_{MS} (2.10) as a function of θ for f_3 at different values of ε.*

We present a sketch of the function f_b for various values of b in Figure 2.1a. For any given positive value of ε, the pulse response function $V_{\mathrm{MS}}(\theta, \varepsilon)$ for $f = f_b$ as in (2.9) is affine:

$$V_{\mathrm{MS}}(\theta, \varepsilon) = m_\varepsilon + K_\varepsilon \theta, \tag{2.10}$$

where $m_\varepsilon = (e^{b\varepsilon} - 1)/(e^b - 1)$ and $K_\varepsilon = e^{b\varepsilon} - 1$. The graph of V_{MS} (2.10) is depicted in Figure 2.1b for different values of ε.

In the numerical computations in this paper, we use the Mirollo-Strogatz model with f_b as in (2.9) with fixed $b = 3$. After fixing b, the parameter space of the system is $\{(\varepsilon, \tau) : \varepsilon > 0, \tau > 0\} = \mathbb{R}^2_+$ where we recall that τ is the *delay* and ε is the *coupling strength*. The qualitative results of our analysis depend only on the properties of the pulse response function V given in definition 2 and not on the specific choice of the Mirollo-Strogatz model (2.8) nor on the choice $f = f_b$ (2.9).

2.1.3 Metric

In this section, we introduce a metric d on $\mathcal{P}_\mathcal{D}$ and thereafter study the continuity of the evolution operator Φ^t with respect to d. Recall that given a phase history function $\phi \in \mathcal{P}^n_\tau$, we can define the extended phase history function ϕ^+.

We define a *lift* (Katok and Hasselblatt 1995) of an extended phase history function ϕ^+ as any function $L_\phi : (-\tau, +\infty) \to \mathbb{R}^n$ such that:

1. $L_\phi(s) \pmod{\mathbb{Z}} = \phi^+(s)$, and

2. For any $s \in (-\tau, +\infty)$ and for $i = 1, \ldots, n$,

$$(L_\phi)_i(s) - (L_\phi)_i(s^-) = \phi^+_i(s) - \phi^+_i(s^-).$$

It follows from these properties that if $L_\phi^{(1)}$ and $L_\phi^{(2)}$ are two lifts of the same extended phase history function ϕ^+ then they differ by a constant integer vector, i.e., $L_\phi^{(1)}(s) - L_\phi^{(2)}(s) = k \in \mathbb{Z}^n$, for all $s \in (-\tau, \infty)$.

Definition 4 (Metric on $\mathcal{P}_\mathcal{D}$). *The metric $d : \mathcal{P}_\mathcal{D} \times \mathcal{P}_\mathcal{D} \to \mathbb{R}$ is given by*

$$d(\phi, \psi) = \min_{k \in \mathbb{Z}^n} \sum_{i=1}^{n} \int_{-\tau}^{\tau} |(L_\phi)_i(s) - (L_\psi)_i(s) - k_i| ds, \tag{2.11}$$

where L_ϕ and L_ψ are arbitrary lifts of ϕ and ψ respectively.

Remark 1. Because of the delay τ, the distance in $\mathbb{T}^n$ between the points $\phi(0)$ and $\psi(0)$ is not a suitable metric for this system. Instead, it is important to take into account the values of ϕ and ψ at least in the interval $(-\tau, 0]$. Nevertheless, if the integral in (2.11) runs from $-\tau$ to 0 defining thus a metric d', there are several states at which the evolution operator is discontinuous with respect to d'. For the chosen metric d in (2.11) the only states for which the evolution operator is discontinuous are those that are related to the *overfiring* effect which is discussed later in this section.

Discontinuity of the evolution operator

In general, the evolution operator $\Phi^t : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{D}$ is not continuous for all $t \geq 0$. We demonstrate this by a simple example. Consider a system of $n = 3$ oscillators and the initial state ϕ given by

$$\begin{aligned} \phi_1(s) = \phi_2(s) &= 1 - \tfrac{1}{2}\tau + s \\ \phi_3(s) &= \vartheta - \tfrac{3}{2}\tau + s \end{aligned}$$

for $s \in (-\tau, 0]$, where $\vartheta \in (0, 1)$ is close enough to 1, so that $H_1(\vartheta) > 1$ which also implies that $H_2(\vartheta) > 1$. Following the rules of definition 3 we extend ϕ to a function ϕ^+ defined on $(-\tau, 2\tau]$. The graphs of ϕ_1^+, ϕ_2^+ and ϕ_3^+ are depicted in Figure 2.2 with the solid lines. Recall that, $\phi^+|_{(-\tau,0]} = \phi$. The most important thing to notice is that the oscillator O_3 receives two simultaneous pulses at $t_f = 3\tau/2$ while $\phi_3^+(t_f^-) = \vartheta$. Therefore, $\phi_3^+(t_f) = \min\{1, H_2(\vartheta)\} \pmod{\mathbb{Z}} = 0$.

Then, consider an initial state ψ given by $\psi_1(s) = \phi_1(s)$, $\psi_2(s) = \phi_2(s) - \epsilon$ and $\psi_3(s) = \phi_3(s)$ for $s \in (-\tau, 0]$, where $\epsilon > 0$ is small. The distance between ϕ and ψ is

$$d(\phi, \psi) = 2\tau\epsilon = O(\epsilon).$$

The graphs of the components ψ_1^+, ψ_2^+ and ψ_3^+ of the extended phase history function ψ^+ are depicted in Figure 2.2 with the dashed lines. The main difference between ϕ and ψ

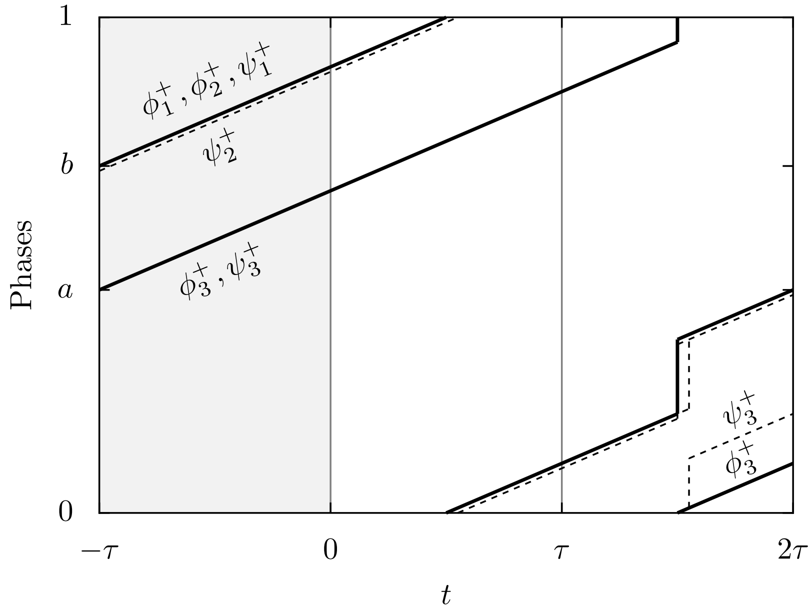

Figure 2.2: *An initial state $\phi \in \mathcal{P}_\mathcal{D}$ for which the map Φ^τ is discontinuous. The graphs of ϕ_j^+, $j = 1, 2, 3$ are represented by the solid lines. The graphs of ψ_j^+, $j = 1, 2, 3$ where ψ^+ is the extended phase history function that corresponds to the initial state ψ are represented by the dashed lines. Recall that ϕ (resp. ψ) is the restriction of ϕ^+ (resp. ψ^+) to $(-\tau, 0]$ (represented by the gray region). The two states diverge abruptly after $t = 3\tau/2$. On the vertical axis, $a = \vartheta - \frac{5}{2}\tau$ and $b = 1 - \frac{3}{2}\tau$.*

is that while in the former the oscillators O_1 and O_2 are synchronized, in the latter they are not. For this reason, the oscillator O_3 receives a single pulse at $t = t_f = 3\tau/2$ while $\psi_3^+(t_f^-) = \vartheta$. Then, $\psi_3^+(t_f) = \min\{1, H_1(\vartheta)\} \pmod{\mathbb{Z}} = 0$, since we assumed before that $H_1(\vartheta) > 1$. O_3 receives a second pulse at $t = t_f + \epsilon$ while $\psi_3^+(t_f + \epsilon^-) = \epsilon$. Hence its phase becomes $\psi_3^+(t_f + \epsilon) = V_1(\epsilon) = V_1(0) + O(\epsilon)$. This would imply that for $s \in (3\tau/2 + \epsilon, 2\tau]$, we have that $\psi_3^+(s) - \phi_3^+(s) = V_1(0) + O(\epsilon)$ where $V_1(0) > 0$ does not depend on ϵ. Then, it is easy to see that

$$d(\Phi^\tau(\psi), \Phi^\tau(\phi)) = \tfrac{1}{2}\tau V_1(0) + O(\epsilon).$$

This shows that the evolution operator Φ^τ is discontinuous at ϕ with respect to the metric d.

We conjecture that the discontinuity of the evolution operator is independent of the choice of a 'reasonable' metric but depends only on the dynamics of the system, and in particular, on the fact that in the example above O_3 fires by receiving two simultaneous pulses but could have fired after receiving a single pulse. Also this discontinuity should not be confused with the fact that the phases of the oscillators are discontinuous functions of time. Motivated by the this discussion we introduce the following definition.

Definition 5. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ and $\theta \in \mathbb{T}$, we define $\nu(\theta)$ as the minimum positive integer for which $H_{\nu(\theta)}(\theta) := H(\theta, \frac{\nu(\theta)}{n-1}\varepsilon) \geq 1$.*

In other words, $\nu(\theta)$ is the minimum number of pulses that will make an oscillator with phase θ to fire. Consider an oscillator whose phase at time t^- is θ and fires after receiving m pulses at t. We say that the oscillator *overfires by $m - \nu(\theta)$ pulses* at t if $\nu(\theta) < m$, i.e., if the oscillator fires after receiving more simultaneous pulses than the strictly necessary number $\nu(\theta)$.

In the context of the example shown in Figure 2.2, we can say that the map $\Phi^\tau : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{D}$ is discontinuous at $\phi \in \mathcal{P}_\mathcal{D}$ because the oscillator O_3 overfires by 1 pulse at $t = t_f = 3\tau/2$.

The existence of discontinuous evolution have also been observed in (Timme 2002, Timme et al. 2002c) where initial states that give such evolution are characterized as *superunstable*.

2.2 Other representations of the dynamics

In this section we present alternative representations of the dynamics. In this section we introduce, following (Ashwin and Timme 2005a), the *past firings* and the *event* representation.

2.2.1 The past firings representation

It follows from definition 3 that the evolution of an initial state $\phi \in \mathcal{P}_\mathcal{D}$ only depends on the values $\phi_i(0)$ and the firing sets $\Sigma_i(\phi)$ that are defined as follows:

Definition 6. *Given a phase history function $\phi \in \mathcal{P}_\mathcal{D}$, the* firing sets $\Sigma_i(\phi) \subset (-\tau, 0]$, *$i = 1, \ldots, n$ are the sets of solutions of the equation $\phi_i(s) = 0$ for $s \in (-\tau, 0]$. The* total firing set *is given by*

$$\Sigma(\phi) = \{(i, \sigma) \,:\, i = 1, \ldots, n, \sigma \in \Sigma_i(\phi), \}$$

Therefore, if we are interested only in the future evolution of the system we can consider the following equivalence relation in $\mathcal{P}_\mathcal{D}$.

Definition 7. *Two phase history functions ϕ_1, ϕ_2 in $\mathcal{P}_\mathcal{D}$ are equivalent, denoted by $\phi_1 \sim \phi_2$, if $\phi_1(0) = \phi_2(0)$ and $\Sigma(\phi_1) = \Sigma(\phi_2)$. Let $\mathbb{P}_\mathcal{D} = \mathcal{P}_\mathcal{D}/\sim$ be the quotient set of equivalence classes and by $[\phi] \in \mathbb{P}_\mathcal{D}$ denote the equivalence class of $\phi \in \mathcal{P}_\mathcal{D}$.*

Points $[\phi] \in \mathbb{P}_\mathcal{D}$ are completely determined by the values of the phases $\phi_i(0)$ and the firing sets $\Sigma(\phi)$ (which may be empty). We denote the elements of $\Sigma_i(\phi)$ by $\sigma_{i,1} > \sigma_{i,2} >$

$\cdots > \sigma_{i,k_i}$ where k_i is the cardinality of $\Sigma_i(\phi)$. Note that by definition, $\phi_i(0) \geq \sigma_{i,1}$, and $\phi_i(0) = 0$ if and only if $\sigma_{i,1} = 0$.

It is possible to give an equivalent description of the dynamics described by definition 3, using only the variables $\phi_i(0)$ and $\sigma_{i,j}$. For such a definition see (Ashwin and Timme 2005a). Notice also that,

Proposition 2. *If $\phi_1 \sim \phi_2$ then*

1. *$\Phi^t(\phi_1) \sim \Phi^t(\phi_2)$ for $t \geq 0$, and*

2. *$\Phi^t(\phi_1) = \Phi^t(\phi_2)$ for $t \geq \tau$.*

Poincaré map

Given a network of n oscillators with dynamics defined by the pulse response function V, with pulse strength ε and with delay τ, we can simplify the study of the system $\mathcal{D} = (n, V, \varepsilon, \tau)$ by considering intersections of the positive semiorbits $\mathcal{O}_+(\phi)$ with the set

$$\mathbf{P} = \{\phi \in \mathcal{P}_{\mathcal{D}} \,:\, \phi_n(0) = 0\}. \tag{2.12}$$

The set $\mathbf{P}$ is called a (Poincaré) *surface of section* (Broer and Takens 2008 (to appear), Strogatz 1994) and it inherits the metric d , see (2.11).

The evolution operator Φ, see (2.6), defines a map $R : \mathbf{P} \to \mathbf{P}$ in the following way. Consider any $\phi \in \mathbf{P}$, i.e., such that $\phi_n(0) = 0$. Since the phases of the oscillators are always increasing there is a minimum time $t(\phi)$ such that the phase of O_n becomes 0 again, i.e., such that $\Phi^{t(\phi)}(\phi)_n(0) = 0$. We define

$$R(\phi) = \Phi^{t(\phi)}(\phi). \tag{2.13}$$

The map R is called *Poincaré (return) map*. Furthermore, we can define the quotient map

$$R_\sim : \mathbf{P}/\sim \to \mathbf{P}/\sim \quad \text{by } [\phi] \mapsto [R(\phi)],$$

of the Poincaré (return) map R, where $\sim$ is the equivalence relation given by definition 7. By Proposition 2 the map $R_\sim$ is well defined.

2.2.2 The event representation

The *event representation* is a symbolic description of the dynamics in which the state of the system is represented by a sequence of events consisting of firings and pulse receptions that would occur. Each event E in the sequence is characterized by a triplet $[K(E), O(E), T(E)]$ where $K(E)$ denotes the type of the event F or mP. The event F denotes a firing event and mP ($m \in \mathbb{N}$) stands for the simultaneous reception of m pulses. The oscillator associated with event $K(E)$ is denoted by $O(E) \in \{1, \dots, n\}$. Finally, $T(E) \in [0, 1]$ denotes how much time is left for the event to occur. For example, the event denoted by $[F, 2, 0.4]$ signifies that the oscillator O_2 will fire after 0.4 time units (and this means that its current phase is $1 - 0.4 = 0.6$), while the event denoted by $[P, 1, 0.3]$ signifies that O_1 is set to receive a pulse after 0.3 time units. We use the shorthand notation $[F, (i_1, \dots, i_k), t]$ and $[mP, (i_1, \dots, i_k), t]$ to indicate that the oscillators $O_{i_1}, \dots, O_{i_k}$ fire or receive m pulses respectively after time t.

Given a particular initial state $\phi \in \mathcal{P}_{\mathcal{D}}$, such that its equivalence class $[\phi] \in \mathbb{P}_{\mathcal{D}}$ is characterized by the phases $\phi_i(0)$ and firing times $\sigma_{i,j}$ for $i = 1, \dots, n$ and $j = 1, \dots, k_i$, consider the space $\mathcal{A}$ of event sequences $\{E_1, E_2, \dots, E_k\}$ of finite (but not fixed) length and the map

$$\mathcal{E} : \mathbb{P}_{\mathcal{D}} \to \mathcal{A} : [\phi] \to \mathcal{E}([\phi]), \tag{2.14}$$

which maps $[\phi]$ to the event sequence $\mathcal{E}([\phi])$ constructed in the following way. First, consider the set Y consisting of the following events:

1. $[F, i, 1 - \phi_i(0)]$ for $i = 1, \dots, n$, and
2. $[P, \ell, \tau + \sigma_{i,j}]$ for $\ell = 1, \dots, n$ with $\ell \neq i$ and $j = 1, \dots, k_i$.

Then, impose time-ordering on Y (i.e., order the events so that events that occur earlier appear first) and in the case that there are $m > 1$ identical events $[P, i, t]$ collect them together to $[mP, i, t]$ to obtain $\mathcal{E}([\phi])$. It follows that $\mathcal{E}$ is injective and hence the inverse map $\mathcal{E}^{-1} : \mathcal{E}(\mathbb{P}_{\mathcal{D}}) \subset \mathcal{A} \to \mathbb{P}_{\mathcal{D}}$ is well defined.

Next, define the map

$$\Phi_{\mathcal{A}} : \mathcal{E}(\mathbb{P}_{\mathcal{D}}) \to \mathcal{E}(\mathbb{P}_{\mathcal{D}}) \tag{2.15}$$

using the following algorithm:

1. For $Z \in \mathcal{E}(\mathbb{P}_{\mathcal{D}})$, consider the first event $E_1 \in Z$ and let $\mathfrak{t} = T(E_1)$. If $T(E_1) \neq 0$ then set $T(E)$ to $T(E) - \mathfrak{t}$ for all $E \in Z$.
2. Take the sequence Z_0 of events $E \in Z$ with $T(E) = 0$ and define $Z_+ = Z \setminus Z_0$. For each event $E \in Z_0$ do the following:
   (a) If $K(E) = F$, then
      i. append to Z_+ the event $[F, O(E), 1]$;
      ii. append to Z_+ the events $[P, \ell, \tau]$ for all $\ell \in \{1, \dots, n\}$ with $\ell \neq O(E)$.
   (b) If $K(E) = mP$, then
      i. find the (unique) event $E' \in Z_+$ with $K(E') = F$ and $O(E') = O(E)$;
      ii. set $T(E')$ to $\max\{T(E') - V(1 - T(E'), m\hat{\varepsilon}), 0\}$.

3. Impose time-ordering on Z_+ and collect together identical pulse events.

4. Set $\Phi_{\mathcal{A}}(Z) = Z_+$.

It follows from the definition of $\Phi_{\mathcal{A}}$ that:

Proposition 3. *1. The map $\Phi_{\mathcal{A}} : \mathcal{E}(\mathbb{P}_{\mathcal{D}}) \to \mathcal{E}(\mathbb{P}_{\mathcal{D}})$ is well defined.*

2. $[\Phi^{\mathsf{t}}(\phi)] = \mathcal{E}^{-1}(\Phi_{\mathcal{A}}(Z))$ where $Z = \mathcal{E}([\phi])$ and t is determined at the first step of the algorithm.

3. Consider an initial state $\phi \in \mathcal{P}_{\mathcal{D}}$ and the corresponding event sequence $\mathcal{E}([\phi])$. If we apply $\Phi_{\mathcal{A}}$, m times to $\mathcal{E}([\phi])$ and the time that elapses at the jth ($j = 1, \ldots, m$) application is t_j with $\mathsf{t} = \sum_j \mathsf{t}_j$, then there exists a unique reconstruction of the extended phase history function ϕ^+ on the interval $[0, \mathsf{t}]$.

The last part of Proposition 3 implies that if $\mathsf{t} \geq \tau$ then it is possible to obtain from the sequence $\{Z, \Phi_{\mathcal{A}}(Z), \Phi^2_{\mathcal{A}}(Z), \ldots, \Phi^m_{\mathcal{A}}(Z)\}$, where $Z = \mathcal{E}([\phi])$, not only the equivalence class $[\Phi^t(\phi)]$ but also the phase history function $\Phi^t(\phi) = \phi^+|_{(t-\tau,t]} \circ T_t$ for any time $t \in [\tau, \mathsf{t}]$.

With this mathematical setting, we prove in chapters 3 and 4 that unstable attractors and heteroclinic cycles exist in an open region (ε, τ) space for a global network of pulse coupled oscillators.

Appendices

2.A Properties of nearby phase history functions

Consider a phase history function $\phi \in \mathcal{P}_{\mathcal{D}}$. Then we show that the characteristics of ϕ determine to a large extent the characteristics of nearby phase history functions $\psi \in \mathcal{P}_{\mathcal{D}}$. In particular, we have the following three Propositions. Notice that in the following we make no distinction between a phase history function $\phi : (-\tau, 0] \to \mathbb{T}^n$ and the corresponding extending phase history function $\phi^+ : (-\tau, \infty) \to \mathbb{T}^n$ and we denote both by ϕ.

Proposition 4. *Assume that ϕ_i has no discontinuities in an interval $(s_1, s_2) \subset [-\tau, \tau]$ and that $\phi_i(s) \neq 0 \pmod{\mathbb{Z}}$ for all $s \in (s_1, s_2)$. Define $E = \frac{1}{8}M(s_2 - s_1)$ where $M = \min\{V_1(0), 1 - \phi_i(s_2)\}$. Then, if $\psi \in \mathcal{P}_{\mathcal{D}}$ satisfies $d(\phi, \psi) = \epsilon < E$ we find that $|\phi_i(s) - \psi_i(s)| < \epsilon_2 = 2\epsilon/(s_2 - s_1)$ for all $s \in (s_1 + \epsilon_1, s_2 - \epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$. In particular, ψ_i has no discontinuities in $(s_1 + \epsilon_1, s_2 - \epsilon_1)$.*

Proof. Assume, for simplicity, that $s_1 = 0$, $s_2 = S < \tau$ and that $\phi_i(s) = u + s$ with $u > 0$ and $u + S < 1$. Then $M = \min\{V_1(0), 1 - (u + S)\}$.

Suppose that $\psi_i(s)$ has one or more discontinuities in $(\epsilon_1, S - \epsilon_1)$ and that one of these discontinuities (caused by $m \geq 1$ simultaneous pulses) is at p.

If $\psi_i(p^+) \geq \phi_i(p)$, then

$$d(\phi, \psi) \geq \int_p^S |\phi_i(s) - \psi_i(s)| ds \geq (\psi_i(p^+) - \phi_i(p))(S - p) \geq (\psi_i(p^+) - \phi_i(p))\epsilon_1$$

The second inequality follows from the fact that ϕ_i increases linearly, while ψ_i increases at least linearly. The third inequality follows from $p < S - \epsilon_1$. Similarly, if $\psi_i(p^-) \leq \phi_i(p)$, then

$$d(\phi, \psi) \geq \int_0^p |\phi_i(s) - \psi_i(s)| ds \geq (\phi_i(p) - \psi_i(p^-))p \geq (\phi_i(p) - \psi_i(p^-))\epsilon_1$$

Again, the second inequality follows from the fact that ϕ_i increases linearly, while ψ_i increases at least linearly. The third inequality follows from $p > \epsilon_1$. Then, we can distinguish three cases.

If both $\psi_i(p^-)$ and $\psi_i(p^+)$ are greater than $\phi_i(p)$, then $\psi_i(p^+) = \min\{1, \psi_i(p^-) + V_m(\psi_i(p^-))\}$. If $\psi_i(p^+) = 1$ then $\psi_i(p^+) - \phi_i(p) = 1 - \phi_i(p) > 1 - (u + S) \geq M$. This means that

$$d(\phi, \psi) \geq M\epsilon_1 = 2\epsilon$$

which is a contradiction. If $\psi_i(p^+) = \psi_i(p^-) + V_m(\psi_i(p^-))$, then $\psi_i(p^+) - \phi_i(p) = V_m(\psi_i(p^-)) + \psi_i(p^-) - \phi_i(p) \geq V_1(0) \geq M$ and we get again a contradiction.

In the second case we assume that both $\psi_i(p^-)$ and $\psi_i(p^+)$ are smaller than $\phi_i(p)$. Then $\phi_i(p) > \psi_i(p^+) = \psi_i(p^-) + V_m(\psi_i(p^-)) > \psi_i(p^-) + V_1(0)$ so we get that $\phi_i(p) - \psi_i(p^-) \geq V_1(0) \geq M$ and

$$d(\phi, \psi) \geq M\epsilon_1 = 2\epsilon.$$

In the third case we assume that $\psi_i(p^-) < \phi_i(p)$ and $\psi_i(p^+) > \phi_i(p)$. This implies that $\phi_i(p) - \psi_i(p^-) = \kappa V_m(\psi_i(p^-)) > \kappa V_1(0) \geq \kappa M$ for some $\kappa \in (0,1)$ and that $\psi_i(p^+) - \phi_i(p) = \min\{1 - \phi_i(p), (1-\kappa)V_m(\psi_i(p^-))\} \geq (1-\kappa)M$. Therefore

$$d(\phi, \psi) \geq \kappa M\epsilon_1 + (1-\kappa)M\epsilon_1 = 2\epsilon.$$

In all cases we have reached a contradiction and this implies that ψ_i can not have any discontinuities in $(\epsilon_1, S - \epsilon_1)$.

This implies also that $\psi_i(s) = u' + s$ for $s \in (\epsilon_1, S - \epsilon_1)$ and in particular that $\phi_i(s) - \psi_i(s) = u - u'$ is constant in this interval. Then, we obtain

$$d(\phi, \psi) \geq \int_{\epsilon_1}^{S-\epsilon_1} |\phi_i(s) - \psi_i(s)| ds = |u - u'|(S - 2\epsilon_1).$$

Hence, $|u - u'| \leq \epsilon/(S - 2\epsilon_1) \leq 2\epsilon/S$. This concludes the proof of the first statement. □

Proposition 5. *Assume that ϕ_i has a discontinuity at $p \in (-\tau, \tau)$, such that $\phi_i(p^+) = H_m(\phi_i(p^-))$ (i.e., the oscillator O_i receives m simultaneous pulses). Also assume that ϕ_i has no other discontinuities in the open interval $(p - \delta, p + \delta)$ and that $\phi_i(s) \notin \mathbb{Z}$ for all $s \in (p - \delta, p + \delta)$. Then, there is $E' > 0$ such that if $\psi \in \mathcal{P}_{\mathcal{D}}$ satisfies $d(\phi, \psi) = \epsilon < E'$ we find that ψ_i receives m pulses in the interval $(p - \epsilon_1, p + \epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$ and $M = \min\{\phi_i(p + \delta), V_1(0)\}$.*

Proof. Since ϕ_i has no discontinuities in $(p - \delta, p)$ and $(p, p + \delta)$ we can apply the previous result in each one of these intervals. Define $M = \min\{1 - \phi_i(p + \delta), V_1(0)\}$. Then for any ψ with $d(\phi, \psi) = \epsilon < \frac{1}{8}M\delta$ we conclude that ψ_i has no discontinuities in the intervals $W_1 = (p - \delta + \epsilon_1, p - \epsilon_1)$ and $W_2 = (p + \epsilon_1, p + \delta - \epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$ and $|\phi_i(s) - \psi_i(s)| < \epsilon_2 = 2\epsilon/\delta$ in the same intervals. Hence,

$$|\psi_i(p - \epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2, \text{ and}$$
$$|\psi_i(p + \epsilon_1) - \phi_i(p^+) - \epsilon_1| < \epsilon_2.$$

Combining the two inequalities we obtain,

$$2(\epsilon_1 - \epsilon_2) < \psi_i(p + \epsilon_1) - \psi_i(p - \epsilon_1) - V_m(\phi_i(p^-)) < 2(\epsilon_1 + \epsilon_2).$$

If ψ_i has discontinuities in $(p - \epsilon_1, p + \epsilon_1)$ that correspond to reception of κ pulses, then

$$\begin{aligned} \psi_i(p - \epsilon_1) + V_\kappa(\psi_i(p - \epsilon_1)) + 2\epsilon_1 &\leq \psi_i(p + \epsilon_1) \\ &\leq \psi_i(p - \epsilon_1) + 2\epsilon_1 + V_\kappa(\psi_i(p - \epsilon_1) + 2\epsilon_1), \end{aligned}$$

or

$$\begin{aligned} V_\kappa(\psi_i(p-\epsilon_1)) + 2\epsilon_1 &\le \psi_i(p+\epsilon_1) - \psi_i(p-\epsilon_1) \\ &\le 2\epsilon_1 + V_\kappa(\psi_i(p-\epsilon_1)) + V_\kappa'(\psi_i(p-\epsilon_1))2\epsilon_1 + O(\epsilon^2). \end{aligned}$$

From $|\psi_i(p-\epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2$, we obtain the estimate:

$$V_\kappa(\phi_i(p^-) - (\epsilon_1+\epsilon_2)) < V_\kappa(\psi_i(p-\epsilon_1)) < V_\kappa(\phi_i(p^-) + (\epsilon_2 - \epsilon_1)),$$

or

$$\begin{aligned} V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\epsilon_1+\epsilon_2) + O(\epsilon^2) \quad &< V_\kappa(\psi_i(p-\epsilon_1)) \\ &< V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\epsilon_2-\epsilon_1) + O(\epsilon^2), \end{aligned}$$

This implies that

$$\begin{gathered} V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\epsilon_1+\epsilon_2) + 2\epsilon_1 + O(\epsilon^2) \le \psi_i(p+\epsilon_1) - \psi_i(p-\epsilon_1) \\ \le 2\epsilon_1 + V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\epsilon_2-\epsilon_1) + V_\kappa'(\psi_i(p-\epsilon_1))2\epsilon_1 + O(\epsilon^2), \end{gathered}$$

or,

$$\begin{aligned} -V_\kappa'(\phi_i(p^-))(\epsilon_1+\epsilon_2) + 2\epsilon_1 + O(\epsilon^2) \quad &\le \psi_i(p+\epsilon_1) - \psi_i(p-\epsilon_1) - V_\kappa(\phi_i(p^-)) \\ &\le 2\epsilon_1 + V_\kappa'(\phi_i(p^-))(\epsilon_1+\epsilon_2) + O(\epsilon^2). \end{aligned}$$

Combining inequalities we obtain

$$\begin{aligned} -V_\kappa'(\phi_i(p^-))(\epsilon_1+\epsilon_2) - 2\epsilon_2 + O(\epsilon^2) \quad &\le V_\kappa(\phi_i(p^-)) - V_m(\phi_i(p^-)) \\ &\le V_\kappa'(\phi_i(p^-))(\epsilon_2+\epsilon_1) + 2\epsilon_2 + O(\epsilon^2) \end{aligned} \tag{2.16}$$

If $\kappa \ne m$ then the difference $|V_\kappa(\phi_i(p^-)) - V_m(\phi_i(p^-))| > |\kappa - m| V_1(0)$ is bounded away from zero. This implies there is some positive $E' < \frac{1}{8} M\delta$ such that the inequality 2.16 does not hold for any $\kappa \ne m$ and $\epsilon < E'$. Therefore, if $\epsilon < E'$ we conclude that $\kappa = m$. This concludes the proof of this part. □

Proposition 6. *Assume that ϕ_i has a discontinuity at $p \in (-\tau, \tau)$, such that $\phi_i(p^+) = 1$ (i.e., the oscillator O_i receives $m \ge \nu(\phi_i(p^+))$ simultaneous pulses and fires). Also assume that ϕ_i has no other discontinuities in the open interval $(p-\delta, p+\delta)$ and that $\phi_i(s) \notin \mathbb{Z}$ for all $s \in (p-\delta, p+\delta)$. Then, there is $E' > 0$ such that if $\psi \in \mathcal{P}_\mathcal{D}$ satisfies $d(\phi, \psi) = \epsilon < E'$ we find that ψ_i receives at least $\nu(\phi_i(p^-))$ pulses in the interval $(p-\epsilon_1, p+\epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$ and $M = \min\{\phi_i(p+\delta), V_1(0)\}$. If ψ_i receives m' pulses with $m' > \nu(\phi_i(p^-))$ then the last $m' - \nu(\phi_i(p^-)) + 1$ pulses are simultaneous.*

Proof. Since ϕ_i has no discontinuities in $(p-\delta,p)$ and $(p,p+\delta)$ we can apply the previous result in each one of these intervals. Define $M = \min\{1-\phi_i(p+\delta), V_1(0)\}$. Then for any ψ with $d(\phi,\psi) = \epsilon < \frac{1}{8}M\delta$ we conclude that ψ_i has no discontinuities in the intervals $W_1 = (p-\delta+\epsilon_1, p-\epsilon_1)$ and $W_2 = (p+\epsilon_1, p+\delta-\epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$ and $|\phi_i(s)-\psi_i(s)| < \epsilon_2 = 2\epsilon/\delta$ in the same intervals. Hence,

$$\begin{aligned}&|\psi_i(p-\epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2, \text{ and}\\ &|\psi_i(p+\epsilon_1) - 1 - \epsilon_1| < \epsilon_2.\end{aligned}$$

Combining the two inequalities we obtain,

$$2(\epsilon_1-\epsilon_2) < \psi_i(p+\epsilon_1) - \psi_i(p-\epsilon_1) - (1-\phi_i(p^-)) < 2(\epsilon_1+\epsilon_2).$$

If ψ_i has discontinuities in $(p-\epsilon_1, p+\epsilon_1)$ that correspond to reception of $\kappa < \nu(\phi_i(p))$ pulses, then

$$\psi_i(p+\epsilon_1) \leq \psi_i(p-\epsilon_1) + 2\epsilon_1 + V_\kappa(\psi_i(p-\epsilon_1)+2\epsilon_1),$$

or

$$\psi_i(p+\epsilon_1) \leq \psi_i(p-\epsilon_1) + 2\epsilon_1 + V_\kappa(\psi_i(p-\epsilon_1)) + V_\kappa'(\psi_i(p-\epsilon_1))2\epsilon_1 + O(\epsilon^2).$$

As in the previous Proposition, since $|\psi_i(p-\epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2$, we obtain the estimate:

$$\begin{aligned}V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\epsilon_1+\epsilon_2) + O(\epsilon^2) &< V_\kappa(\psi_i(p-\epsilon_1))\\ &< V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\epsilon_2-\epsilon_1) + O(\epsilon^2),\end{aligned}$$

Therefore

$$\psi_i(p+\epsilon_1) \leq H_\kappa(\phi_i(p^-)) + \epsilon_2 + \epsilon_1 + V_\kappa'(\phi_i(p^-))(\epsilon_2-\epsilon_1) + V_\kappa'(\psi_i(p-\epsilon_1))2\epsilon_1 + O(\epsilon^2).$$

Since $\kappa < \nu(\phi_i(p))$ we deduce that by taking ϵ small enough we could have $\psi_i(p+\epsilon_1) < 1$ which is a contradiction. This implies that $\kappa \geq \nu(\phi_i(p^-))$. Moreover, if $\kappa > \nu(\phi_i(p^-))$ then the last $\kappa - \nu(\phi_i(p^-)) + 1$ pulses must be simultaneous, otherwise $\psi_i(p+\epsilon_1) > H_1(0)$ which gives a contradiction since $\psi_i(p+\epsilon_1) = O(\epsilon)$. □